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A Constructive Analysis of the Remes Algorithm

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1. INTRODUCTION

Numerical analysis is one branch of mathematics in which we should expect considerable dividends from the application of the rigorously constructive methods advocated by Bishop [1]. Indeed, even if one does not agree with Bishop's bold statement that

> Every theorem proved with idealistic methods presents a challenge: to find a constructive version and to give it a constructive proof [1, p. 9].

there is a very strong argument for assent to this dictum in the case of theorems of numerical analysis.

Consider, for example, the well-known algorithm of Remes for the computation of best Chebyshev approximants over [0, 1]. With minor modifications, the procedure of this algorithm can be made constructive in our (that is, Bishop's) sense: in particular, we must avoid any appeal to the theorem which asserts the attainment of the supremum of a continuous, real-valued mapping on [0, 1] (cf. [2, p. 4]). What has not been given until now is a constructive proof of the convergence of this algorithm.

Here we have the remarkable situation where an important, practical algorithm has no known rate of convergence in general! We believe that this state of affairs should never arise in numerical analysis: in our view, an algorithm is incomplete unless it is presented along with a constructive proof of its convergence. In keeping with this belief, the main aim of this paper is to produce a constructive proof of the convergence of the Remes algorithm.

Throughout our work, $C[0, 1]$ will be the real linear space (with the usual "sup" norm) of continuous mappings of $[0, 1]$ into $\mathbb{R}$, n a positive integer, and $\{\phi_1,..., \phi_n\}$ a subset of $C[0, 1]$ which satisfies the *Haar condition*:

if $K_1,..., K_n$ are pairwise disjoint compact subsets of $[0, 1]$, *then*

$$\inf\{|\det[\phi_j(x_i)]|: \forall i\ (x_i \in K_i)\} > 0.$$

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(Note that, for constructive purposes, K_i, K_j are *disjoint* if $\inf\{|x_i - x_j|: x_i \in K_i, x_j \in K_j\} > 0$. The constructive Haar condition is discussed in Section 1 of [4].) We say that $\{\phi_1, \ldots, \phi_n\}$ is a *Chebyshev system* over $[0, 1]$, and denote by H the subspace of $C[0, 1]$ which it spans.

Given $\mathbf{a} = (a_1, \ldots, a_m)$ in $\mathbb{R}^m$, we write $\|\mathbf{a}\|_2$ for $(\sum_{i=1}^m a_i^2)^{1/2}$. We also write

$$\boldsymbol{\phi}(x) = (\phi_1(x), \ldots, \phi_n(x)) \qquad (x \in [0, 1])$$

and

$$\|\boldsymbol{\phi}\| = \sup\{\|\boldsymbol{\phi}(x)\|_2 : x \in [0, 1]\}$$

(the latter being computable, by [2, Chap. 2, 4.4]). Note that, by the Cauchy–Schwarz inequality,

$$\left| \sum_{i=1}^n a_i(\phi_i(x) - \phi_i(y)) \right| \leqslant \|\mathbf{a}\|_2 \, \|\boldsymbol{\phi}(x) - \boldsymbol{\phi}(y)\|_2$$

for all x, y in $[0, 1]$.

We assume familiarity with a general background in constructive analysis (as found in [1] or [2]), and with the constructive development of Chebyshev approximation theory [3, 4]. However, it is convenient to be reminded of the following particular case of [4, Proposition 1.1]:

1.1. PROPOSITION. *Let $n \geqslant 2$ and $0 < \alpha \leqslant n^{-1}$. Then there exists $\beta > 0$ such that $|\det[\phi_j(x_i)]| \geqslant \beta$ whenever $x_1, \ldots, x_n$ belong to $[0, 1]$ and $\min_{1 \leqslant i < j \leqslant n} |x_i - x_j| \geqslant \alpha$.* ∎

We now introduce two important mappings β, γ: $(0, 1/n] \to \mathbb{R}^+$. For each $\alpha \in (0, 1/n]$, define

$$\begin{aligned} \beta(\alpha) &= \inf\{|\phi_1(x)| : x \in [0, 1]\} && \text{if} \quad n = 1, \\ &= \inf\{|\det[\phi_j(x_i)]| : x_1, \ldots, x_n \in [0, 1], \\ &\qquad \min_{1 \leqslant i < j \leqslant n} |x_i - x_j| \geqslant \alpha\} && \text{if} \quad n \geqslant 2, \end{aligned}$$

and

$$\gamma(\alpha) = \min \left(\|\boldsymbol{\phi}\|, \beta(\alpha)/n^{1/2}(n-1)! \prod_{i=1}^n (1 + \|\phi_i\|) \right).$$

In the case $n \geqslant 2$, $\beta(\alpha)$ is well defined as the mapping $\mathbf{x} \to |\det[\phi_j(x_i)]|$ is continuous on the compact set $\{\mathbf{x} \in [0, 1]^n : \min_{1 \leqslant i < j \leqslant n} |x_i - x_j| \geqslant \alpha\}$.

2. Some Auxiliary Results

The results below are needed for our discussion of the Remes Algorithm in Sections 3 and 4 below.

2.1. Lemma. *Let* $0 < \alpha \leqslant n^{-1}$, *and let* $x_1, \ldots, x_{n+1}$ *be points of* $[0, 1]$ *with* $\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant \alpha$. *Let* $\lambda_1, \ldots, \lambda_{n+1}$ *be real numbers such that* $\lambda_1 > 0$, $\sum_{i=1}^{n+1} |\lambda_i| = 1$ *and* $\sum_{i=1}^{n+1} \lambda_i \phi(x_i) = \mathbf{0}$. *Then*

$$(-1)^{r-1} \lambda_r \geqslant (\gamma(\alpha)/\|\phi\|)^{r-1} \Big/ \sum_{k=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{k-1}$$

for each r *in* $\{1, \ldots, n+1\}$.

Proof. By [4, 4.1],

$$(\gamma(\alpha)/\|\phi\|)^{k-1} \leqslant (-1)^{k-1} \lambda_1^{-1} \lambda_k \leqslant (\|\phi\|/\gamma(\alpha))^{k-1}$$

for $k = 1, \ldots, n+1$; so that

$$\begin{aligned} 0 < \lambda_1^{-1} &= \sum_{k=1}^{n+1} \lambda_1^{-1} |\lambda_k| \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} \lambda_1^{-1} \lambda_k \\ &\leqslant \sum_{k=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{k-1}. \end{aligned}$$

For each r in $\{1, \ldots, n+1\}$, we now have

$$\begin{aligned} (-1)^{r-1} \lambda_r &\geqslant \lambda_1 (\gamma(\alpha)/\|\phi\|)^{r-1} \\ &\geqslant (\gamma(\alpha)/\|\phi\|)^{r-1} \Big/ \sum_{k=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{k-1}. \quad \blacksquare \end{aligned}$$

Let $(x_1, \ldots, x_{n+1})$ be a strictly increasing sequence of $n+1$ points of $[0, 1]$, and $\lambda_1, \ldots, \lambda_{n+1}$ real numbers such that $\sum_{i=1}^{n+1} |\lambda_i| = 1$ and $\sum_{i=1}^{n+1} \lambda_i \phi(x_i) = \mathbf{0}$. Then (as we leave the reader to prove) $|\lambda_i| > 0$ for each i (cf. [4, 4.1]).

Our only application of the next result will be in the proof of a partial converse to 2.1. However, we regard 2.2 as of considerable interest in its own right, and are surprised to find no reference to it in the classical literature.

2.2. Proposition. *Let* $a \in C[0, 1]$. *Then there exist operations* $\delta: \mathbb{R}^+ \to \mathbb{R}^+$, $\omega: \mathbb{R}^+ \to \mathbb{R}^+$ *(depending on* a*) such that: if* $\varepsilon > 0$, *and*

$x_1, \ldots, x_{n+1}$ are points of $[0, 1]$ with $0 < \min_{k=1,\ldots,n}(x_{k+1} - x_k) \leqslant \delta(\varepsilon)$, then there exists h in H with modulus of continuity ω, such that $\max_{k=1,\ldots,n+1} |(a-h)(x_k)| \leqslant \varepsilon$.

Proof. It will suffice to take $m = n + 1$ in the following lemma.

2.2.1. *Let $m \in \{2,\ldots, n+1\}$. There exist operations $\delta_m: \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_m: \mathbb{R}^+ \to \mathbb{R}^+$ such that: if $\varepsilon > 0$, and $x_1, \ldots, x_m$ are points of $[0, 1]$ with $0 < \min_{k=1,\ldots,m-1}(x_{k+1} - x_k) \leqslant \delta_m(\varepsilon)$, then there exists h_m in H with modulus of continuity ω_m, such that $\max_{k=1,\ldots,m} |(a - h_m)(x_k)| \leqslant \varepsilon$.*

We prove 2.2.1 by induction on m. If $m = 2$, we choose ψ in H so that

$$0 < \mu = \inf\{\psi(x) : x \in [0, 1]\} \leqslant \|\psi\| \leqslant 1$$

[4, 2.7]. With ω_1 a common modulus of continuity for a and ψ on $[0, 1]$, we define

$$\delta_2(\varepsilon) = \omega_2(\varepsilon) = \omega_1(\mu\varepsilon/2(1 + \|a\|)) \qquad (\varepsilon > 0).$$

Let $\varepsilon > 0$, $0 \leqslant x_1 < x_2 \leqslant 1$ and $x_2 - x_1 \leqslant \delta_2(\varepsilon)$. With $h_2 = a(x_1)\,\psi(x_1)^{-1}\,\psi$, we then have $h_2 \in H$, $h_2(x_1) = a(x_1)$ and

$$\begin{aligned} |(a - h_2)(x_2)| &\leqslant \psi(x_1)^{-1}\,|a(x_1)\,\psi(x_2) - a(x_2)\,\psi(x_1)| \\ &\leqslant \mu^{-1}(|a(x_1)|\,|\psi(x_2) - \psi(x_1)| + \psi(x_1)\,|a(x_2) - a(x_1)|) \\ &\leqslant \mu^{-1}(\|a\|\,|\psi(x_2) - \psi(x_1)| + |a(x_2) - a(x_1)|) \\ &\leqslant \varepsilon. \end{aligned}$$

Moreover, if $\varepsilon' > 0$, x and y belong to $[0, 1]$ and $|x - y| \leqslant \omega_2(\varepsilon')$, then

$$\begin{aligned} |h_2(x) - h_2(y)| &= \psi(x_1)^{-1}\,|a(x_1)|\,|\psi(x) - \psi(y)| \\ &\leqslant \mu^{-1}\,\|a\|\,|\psi(x) - \psi(y)| \\ &\leqslant \varepsilon'. \end{aligned}$$

This completes the proof in the case $m = 2$.

Now let $k \in \{2,\ldots, n\}$, suppose we have proved 2.2.1 for $m = k$, and consider the case $m = k + 1$. Let ω be a common modulus of continuity for a and ϕ on $[0, 1]$, and, for each $\varepsilon > 0$, define

$$\begin{aligned} \omega_{k+1}(\varepsilon) &= \min(\omega_k(\varepsilon), \omega(\varepsilon\gamma(a)/n(1 + \|a\|))), \\ \delta_{k+1}(\varepsilon) &= \tfrac{1}{2}\min(\omega(\varepsilon/3), \omega_{k+1}(\varepsilon/3)). \end{aligned}$$

Given $\varepsilon > 0$, let $x_1, \ldots, x_{k+1}$ be points of $[0, 1]$ with

$$0 < \min_{j=1,\ldots,k} (x_{j+1} - x_j) \leqslant \delta_{k+1}(\varepsilon),$$

and choose r in $\{1,\ldots,k\}$ so that $x_{r+1}-x_r<2\delta_{k+1}(\varepsilon)$. Let $(x'_1,\ldots,x'_k)$ be a rearrangement of the numbers x_j $(j\neq r)$ into a strictly increasing sequence, $\sigma=\min_{j=1,\ldots,k-1}(x'_{j+1}-x'_j)$ and $\alpha=(2n)^{-1}\delta_k(\varepsilon/3)$. Either $2n\alpha>\sigma$ or $\sigma>n\alpha$. In the former case, $\sigma<\delta_k(\varepsilon/3)$, and so we can find h_k in H, with modulus of continuity ω_k, such that

$$|(a-h_k)(x'_j)|\leqslant \varepsilon/3 \qquad (j=1,\ldots,k).$$

Then

$$\begin{aligned}|(a-h_k)(x_r)| &\leqslant |a(x_{r+1})-a(x_r)|+|(a-h_k)(x_{r+1})|\\ &\quad +|h_k(x_{r+1})-h_k(x_r)|\\ &<\varepsilon/3+\varepsilon/3+\varepsilon/3\\ &=\varepsilon.\end{aligned}$$

Thus

$$|(a-h_k)(x_j)|<\varepsilon \qquad (j=1,\ldots,k+1).$$

As $\omega_{k+1}\leqslant\omega_k$, h_k has modulus of continuity ω_{k+1}. We therefore need only set $h_{k+1}=h_k$.

In the case $\sigma>n\alpha$, we choose $\xi_1,\ldots,\xi_n$ in $[0,1]$ so that

$$x_j\in\{\xi_1,\ldots,\xi_n\} \qquad (j\in\{1,\ldots,k+1\}, j\neq r)$$

and $\min_{j=1,\ldots,n-1}(\xi_{j+1}-\xi_j)\geqslant\alpha$. We then construct $h_{k+1}=\sum_{i=1}^n c_i\phi_i$ in H so that $h_{k+1}(\xi_j)=a(\xi_j)$ for each j in $\{1,\ldots,n\}$. (This construction is possible in view of the Haar condition.) By [4, 2.1], we have

$$\|\mathbf{c}\|_2\leqslant n\gamma(\alpha)^{-1}\max_{j=1,\ldots,n}|h_{k+1}(\xi_j)|\leqslant n\gamma(\alpha)^{-1}\|a\|,$$

where $\mathbf{c}=(c_1,\ldots,c_n)$. Hence

$$|h_{k+1}(x)-h_{k+1}(y)|\leqslant n\gamma(\alpha)^{-1}\|a\|\,\|\boldsymbol{\phi}(x)-\boldsymbol{\phi}(y)\|_2$$

for all x, y in $[0,1]$. It is now clear that h_{k+1} has modulus of continuity ω_{k+1}. On the other hand, by our choice of r,

$$\begin{aligned}|(a-h_{k+1})(x_r)| &\leqslant |a(x_{r+1})-a(x_r)|+|(a-h_{k+1})(x_{r+1})|\\ &\quad +|h_{k+1}(x_{r+1})-h_{k+1}(x_r)|\\ &\leqslant \varepsilon/3+0+n\gamma(\alpha)^{-1}\|a\|\,\|\boldsymbol{\phi}(x_{r+1})-\boldsymbol{\phi}(x_r)\|_2\\ &<\varepsilon.\end{aligned}$$

This completes our inductive proof. ∎

2.3. LEMMA. *Let $a \in C[0,1]$, and let δ be as in 2.2. Let $(x_1, \ldots, x_{n+1})$ be a strictly increasing sequence of $n+1$ points of $[0,1]$, and $c, \lambda_1, \ldots, \lambda_{n+1}$ real numbers such that $\sum_{i=1}^{n+1} |\lambda_i| = 1$, $\sum_{i=1}^{n+1} \lambda_i \boldsymbol{\phi}(x_i) = \mathbf{0}$ and $|\sum_{i=1}^{n+1} \lambda_i a(x_i)| \geqslant 2c > 0$. Then $\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant \delta(c)$.*

Proof. Suppose that $\min_{k=1,\ldots,n}(x_{k+1} - x_k) < \delta(c)$, and construct $h = \sum_{i=1}^{n} h_i \phi_i$ in H so that $\max_{k=1,\ldots,n+1} |(a-h)(x_k)| \leqslant c$. Then

$$\sum_{i=1}^{n+1} \lambda_i h(x_i) = \sum_{j=1}^{n} h_j \sum_{i=1}^{n+1} \lambda_i \phi_j(x_i) = 0,$$

and so

$$\begin{aligned} c &\leqslant 2^{-1} \left| \sum_{i=1}^{n+1} \lambda_i (a-h)(x_i) \right| \\ &\leqslant 2^{-1} \sum_{i=1}^{n+1} |\lambda_i| \, |(a-h)(x_i)| \\ &\leqslant 2^{-1} \sum_{i=1}^{n+1} |\lambda_i| \, c \\ &= c/2, \end{aligned}$$

a contradiction. Hence, in fact, $\min_{k=1,\ldots,n}(x_{k+1} - x_k) \geqslant \delta(c)$. ∎

2.4. LEMMA. *Let $0 \leqslant x_1 < \cdots < x_{n+1} \leqslant 1$, and let $\lambda_1, \ldots, \lambda_{n+1}$ be real numbers such that $\sum_{i=1}^{n+1} |\lambda_i| = 1$, $\sum_{i=1}^{n+1} \lambda_i \boldsymbol{\phi}(x_i) = \mathbf{0}$. Then*

$$\inf_{h \in H} \max_{k=1,\ldots,n+1} |(a-h)(x_k)| = \left| \sum_{i=1}^{n+1} \lambda_i a(x_i) \right|.$$

Proof. Let $b_0 \in H$ and $c \in \mathbb{R}$ satisfy the equations

$$b_0(x_i) + (-1)^i c = a(x_i) \qquad (i = 1, \ldots, n+1).$$

Then [4, 5.1], for each i,

$$\begin{aligned} \inf_{h \in H} \max_{k=1,\ldots,n+1} |(a-h)(x_k)| &= |(a-b_0)(x_i)| \\ &= |c|. \end{aligned}$$

Noting that $(-1)^{i-1} \lambda_i > 0$ for each i (2.1), and that $\sum_{i=1}^{n+1} \lambda_i h(x_i) = 0$ for each h in H, we see that

$$\begin{aligned} c &= \sum_{i=1}^{n+1} |\lambda_i| c \\ &= -\sum_{i=1}^{n+1} \lambda_i(-1)^i c \\ &= -\sum_{i=1}^{n+1} \lambda_i(a - b_0)(x_i) \\ &= -\sum_{i=1}^{n+1} \lambda_i a(x_i). \end{aligned}$$

The result now follows. ∎

3. The Constructive Remes Algorithm

Throughout the remaining two sections, a will be a fixed element of $C[0, 1]$ such that

$$0 < d = \text{dist}(a, H) = \inf\{\|a - h\| : h \in H\},$$

and b will be the best approximant of a in H. That b is computable is shown in 4.5 of [4]. We wish to discuss the convergence of the *constructive Remes Algorithm*, the procedure of which we now describe.

Choose a strictly increasing sequence $(x_{1,1},..., x_{1,n+1})$ of $n + 1$ points of $[0, 1]$ such that

$$0 < d_1 = \inf_{h \in H} \max_{k=1,...,n+1} |(a - h)(x_{1,k})|.$$

(We shall discuss the mechanism of this choice in Section 4 below.) Set

$$\alpha = \min(n^{-1}, \delta(d_1/2)),$$

$$\kappa = \frac{1}{2} (\gamma(\alpha)/\|\phi\|)^n \Big/ \sum_{i=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{i-1},$$

with δ as in 2.2 above. Note that $0 < \kappa < 1$, by 2.1.

Choose also a strictly decreasing sequence $(s_n)_{n \geqslant 1}$ of positive numbers converging to 0, such that $s_1 < d$. Construct recursively sequences λ in $\{0, 1\}$, $((x_{\nu,1},..., x_{\nu,n+1}))_{\nu \geqslant 1}$ in $[0, 1]^{n+1}$, $(d_\nu)_{\nu \geqslant 1}$ in $\mathbb{R}^+$, and $(b_\nu)_{\nu \geqslant 1}$ in H, so that

(i) $0 \leqslant x_{\nu,1} < \cdots < x_{\nu,n+1} \leqslant 1$,

(ii) $0 < d_\nu = \inf_{h \in H} \max_{k=1,\ldots,n+1} |(a-h)(x_{\nu,k})| = \max_{k=1,\ldots,n+1} |(a-b_\nu)(x_{\nu,k})|$,

(iii) $\lambda(\nu) = 0 \Rightarrow d_\nu > \|a - b_\nu\| - s_\nu$,

(iv) $\lambda(\nu) = 1 \Rightarrow \|a - b_\nu\| > d_\nu$,

(v) $\lambda(\nu) = 0 \Rightarrow x_{\nu+1,k} = x_{\nu,k}$ for each k in $\{1,\ldots, n+1\}$,

(vi) $\lambda(\nu) = 1 \Rightarrow$ there exist integers $k(\nu) \in \{1,\ldots, n+1\}$ and $j(\nu) \in \{0, 1\}$ such that

$$x_{\nu+1,i} \in \{x_{\nu,r}: r = 1,\ldots, n+1\} \qquad \text{for each } i \neq k(\nu),$$

$$|(a-b_\nu)(x_{\nu+1,k(\nu)})| > \tfrac{1}{2}(\|a-b_\nu\| + d_\nu)$$

and

$$(-1)^{i+j(\nu)}(a-b_\nu)(x_{\nu+1,i}) > 0 \qquad \text{for each } i.$$

To show that this construction is possible, suppose that we have constructed $\lambda(\nu)$, $(x_{\nu,1},\ldots, x_{\nu,n+1})$, d_ν and b_ν satisfying (i)–(iv). If $\lambda(\nu) = 0$, set $x_{\nu+1,k} = x_{\nu,k}$ $(k = 1,\ldots, n+1)$, $d_{\nu+1} = d_\nu$ and $b_{\nu+1} = b_\nu$. If $\lambda(\nu) = 1$, first observe that, as $d_\nu > 0$, there exists $r \in \{0, 1\}$ such that

$$(-1)^{k-r}(a-b_\nu)(x_{\nu,k}) = d_\nu \qquad (k = 1,\ldots, n+1)$$

[4, 5.2 and the remarks following it]. Applying the method of exchange to the numbers $x_{\nu,1},\ldots, x_{\nu,n+1}$ and ξ [5, pp. 107, 108], we can find a strictly increasing sequence $(x_{\nu+1,1},\ldots, x_{\nu+1,n+1})$ in $[0, 1]$, and integers $k(\nu) \in \{1,\ldots, n+1\}$, $j(\nu) \in \{0, 1\}$, with the properties described under (vi) above. Let

$$d_{\nu+1} = \inf_{h \in H} \max_{k=1,\ldots,n+1} |(a-h)(x_{\nu+1,k})|,$$

and construct $b_{\nu+1} \in H$ so that

$$\max_{k=1,\ldots,n+1} |(a-b_{\nu+1})(x_{\nu+1,k})| = d_{\nu+1}$$

[4, 5.1]. We must show that $d_{\nu+1} > 0$. To do so, choose real numbers $\lambda_1,\ldots, \lambda_{n+1}$ so that $\sum_{i=1}^{n+1} |\lambda_i| = 1$ and $\sum_{i=1}^{n+1} \lambda_i \phi(x_{\nu+1,i}) = 0$. (This is possible as $\{\phi_1,\ldots, \phi_n\}$ satisfies the Haar condition.) Note that, for $i = 1,\ldots, n+1$,

$$\begin{aligned}(-1)^{j(\nu)+1} \lambda_i (a-b_\nu)(x_{\nu+1,i}) &= (-1)^{i-1} \lambda_i (-1)^{i+j(\nu)}(a-b_\nu)(x_{\nu+1,i}) \\ &= |\lambda_i|\, |(a-b_\nu)(x_{\nu+1,i})|.\end{aligned}$$

It follows from this and 2.4 that

$$\begin{aligned} d_{\nu+1} &= \left| \sum_{i=1}^{n+1} \lambda_i a(x_{\nu+1,i}) \right| \\ &= \left| \sum_{i=1}^{n+1} (-1)^{j(\nu)+1} \lambda_i (a-b_\nu)(x_{\nu+1,i}) \right| \\ &= \sum_{i=1}^{n+1} |\lambda_i| \, |(a-b_\nu)(x_{\nu+1,i})| \\ &= d_\nu + \sum_{i=1}^{n+1} |\lambda_i| \, (|(a-b_\nu)(x_{\nu+1,i})| - d_\nu). \end{aligned}$$

Now, if $i \in \{1,..., n+1\}$ and $i \neq k(\nu)$, there exists j with $x_{\nu+1,i} = x_{\nu,j}$. In that case, by [4, 5.1], $|(a-b_\nu)(x_{\nu+1,i})| = d_\nu$. On the other hand, by definition of $k(\nu)$,

$$|(a-b_\nu)(x_{\nu+1,k(\nu)})| - d_\nu > \tfrac{1}{2}(\|a-b_\nu\| - d_\nu) > 0.$$

Hence

$$d_{\nu+1} > d_\nu + \tfrac{1}{2} |\lambda_{k(\nu)}| \, (\|a-b_\nu\| - d_\nu), \tag{3.1}$$

and so $d_{\nu+1} > d_\nu > 0$.

We complete the inductive construction as follows. Either $d_{\nu+1} > \|a-b_{\nu+1}\| - s_{\nu+1}$, in which case we set $\lambda(\nu+1) = 0$; or $\|a-b_{\nu+1}\| > d_{\nu+1}$, when we set $\lambda(\nu+1) = 1$.

Note that the dichotomy

$$\|a-b_{\nu+1}\| > d_{\nu+1} \qquad \text{or} \qquad d_{\nu+1} > \|a-b_{\nu+1}\| - s_{\nu+1}$$

is introduced here because the classical proposition

$$\forall x \in \mathbb{R} \qquad (x \geqslant 0 \Rightarrow x > 0 \text{ or } x = 0)$$

is essentially nonconstructive [1, p. 26].

We now derive a series of results about the convergence of d_ν to d and of b_ν to b.

3.2. Lemma. *For each integer* $\nu \geqslant 1$, $d_\nu \leqslant d_{\nu+1} \leqslant d$. *Moreover, if* $\lambda(\nu) = 1$, *then* $d_{\nu+1} > d_\nu + \kappa(\|a-b_\nu\| - d_\nu)$.

Proof. We have

$$\begin{aligned} d_\nu &= \inf_{h \in H} \max_{k=1,\ldots,n+1} |(a-h)(x_{\nu,k})| \\ &\leqslant \max_{k=1,\ldots,n+1} |(a-b)(x_{\nu,k})| \\ &\leqslant \|a-b\| \\ &= d. \end{aligned}$$

On the other hand, $d_{\nu+1} = d_\nu$ when $\lambda(\nu) = 0$; while if $\lambda(\nu) = 1$, then, by 3.1 above,

$$d_{\nu+1} > d_\nu + \tfrac{1}{2} |\lambda_{k(\nu)}| (\|a - b_\nu\| - d_\nu) > d_\nu.$$

Moreover, in the latter case we have

$$0 < d_1 < d_{\nu+1} = \left| \sum_{i=1}^{n+1} \lambda_i a(x_{\nu+1,i}) \right|,$$

so that (by 2.3)

$$\min_{k=1,\ldots,n} (x_{\nu+1,k+1} - x_{\nu+1,k}) \geqslant \delta(d_1/2) \geqslant \alpha.$$

Thus, by 2.1,

$$|\lambda_{k(\nu)}| \geqslant (\gamma(\alpha)/\|\phi\|)^{\nu-1} \Big/ \sum_{k=1}^{n+1} (\|\phi\|/\gamma(\alpha))^{k-1} \geqslant 2\kappa,$$

and so $d_{\nu+1} > d_\nu + \kappa(\|a - b_\nu\| - d_\nu)$. ∎

3.3. LEMMA. *If* $\lambda(\nu) = 1$, *then* $d - d_{\nu+1} < (1-\kappa)(d - d_\nu)$.

Proof. As $d = \operatorname{dist}(a, H) \leqslant \|a - b_\nu\|$, it follows from 3.2 that, if $\lambda(\nu) = 1$,

$$\begin{aligned} d_{\nu+1} &> d_\nu + \kappa(\|a - b_\nu\| - d_\nu) \\ &\geqslant d_\nu + \kappa(d - d_\nu), \end{aligned}$$

whence $d - d_{\nu+1} < (1-\kappa)(d - d_\nu)$. ∎

Before we produce our convergence estimates, it will be helpful to introduce the numbers $c_1 = d - d_1$ and

$$c_n = \max((1-\kappa)^{n-1}(d - d_1), \max_{j=1,\ldots,n-1} (1-\kappa)^{j-1} s_{n-j}) \qquad (n \geqslant 2).$$

Note that

$$c_{n+1} = \max((1-\kappa)\,c_n, s_n) \qquad (n \geqslant 1)$$

and that the sequence $(c_n)_{n \geqslant 1}$ of positive numbers converges to 0.

3.4. THEOREM. *For each integer* $\nu \geqslant 1$, $0 \leqslant d - d_\nu \leqslant c_\nu$.

Proof. In view of 3.2, we need only prove that $d - d_\nu \leqslant c_\nu$. This is trivial for $\nu = 1$. Suppose we have proved it for $\nu = N$, and consider the case $\nu = N+1$. If $\lambda(N) = 0$, then

$$d_{N+1} = d_N > \|a - b_N\| - s_N \geqslant d - s_N,$$

and so

$$d - d_{N+1} < s_N \leqslant c_{N+1}.$$

If $\lambda(N) = 1$, then

$$\begin{aligned} d - d_{N+1} &< (1-\kappa)(d - d_N) \\ &\leqslant (1-\kappa)\,c_N \\ &\leqslant c_{N+1}. \end{aligned}$$

This completes the induction. ∎

3.5. THEOREM. *Let* ω *be a modulus of continuity for* $a - b$ *on* $[0, 1]$, $t = \min(n^{-1}, \omega(\|a - b\|))$ *and* $A = n^2(\|\phi\|/\gamma(t))^{2n+1}$. *Then*

$$\|b_\nu - b\| \leqslant A \max(s_\nu, \kappa^{-1} c_\nu)$$

for each integer $\nu \geqslant 1$.

Proof. If $\lambda(\nu) = 0$, then

$$\|a - b_\nu\| - \|a - b\| \leqslant \|a - b_\nu\| - d_\nu < s_\nu.$$

If $\lambda(\nu) = 1$, then (3.2)

$$\begin{aligned} d_{\nu+1} &> d_\nu + \kappa(\|a - b_\nu\| - d_\nu) \\ &\geqslant d_\nu + \kappa(\|a - b_\nu\| - \|a - b\|); \end{aligned}$$

so that, by 2.4 and 3.4,

$$\begin{aligned} \|a - b_\nu\| - \|a - b\| &< \kappa^{-1}(d_{\nu+1} - d_\nu) \\ &\leqslant \kappa^{-1}(d - d_\nu) \\ &\leqslant \kappa^{-1} c_\nu. \end{aligned}$$

Now, the Strong Unicity Theorem [4, 4.6] tells us that

$$\|a - p\| \geqslant \|a - b\| + A^{-1} \|p - b\| \qquad (p \in H).$$

Hence

$$\begin{aligned} \|b_v - b\| &\leqslant A(\|a - b_v\| - \|a - b\|) \\ &\leqslant A \max(s_v, \kappa^{-1} c_v). \quad \blacksquare \end{aligned}$$

It must be stressed that we make no claim for the efficiency of the estimates in 3.4 and 3.5 above. Indeed, we can hardly expect such estimates to be efficient, as they cover even the most pathological examples of Chebyshev approximation. In practice, κ tends to be very close to 0: for example, if $n = 2$, $\phi_1(x) = 1$, $\phi_2(x) = x$ and $a(x) = x^2$, we find that $\kappa < 10^{-16}$.

In 3.5, our ability to estimate $\|b_v - b\|$ depends on our knowledge of a modulus of continuity for $a - b$. That such knowledge can be obtained without prior computation of b is shown at the end of [4]. In fact, if $x_i = i/n$ $(i = 1,..., n)$, $\Delta = \det[\phi_j(x_i)]$,

$$c = 2 \|a\| \, |\Delta|^{-1} n^{3/2} (n - 1)! \prod_{r=1}^{n} (1 + \|\phi_r\|)$$

and ω_1 is a modulus of continuity for $\boldsymbol{\phi}$ on $[0, 1]$, then $|b(x) - b(y)| \leqslant \varepsilon$ whenever x, y belong to $[0, 1]$ and $|x - y| \leqslant \omega_1(\varepsilon/c)$.

4. The First Step

We close our discussion of the Remes algorithm by showing how to carry out its first step.

To prove that this can be done in theory is quite simple. By [4, 3.4], there exist $j \in \{0, 1\}$ and a strictly increasing sequence $(x_1,..., x_{n+1})$ of $n + 1$ points of $[0, 1]$ such that

$$(-1)^{k-j}(a - b)(x_k) > d - d/2(n + 1).$$

Construct $b_1 \in H$ so that

$$|(a - b_1)(x_k)| = \inf_{h \in H} \max_{r = 1,...,n+1} |(a - h)(x_r)| \qquad (k = 1,..., n + 1)$$

[4, 5.1]. Then for each $k \in \{1,..., n + 1\}$, we have $|(b - b_1)(x_k)| \leqslant nd/2(n + 1)$ [4, 5.3], and therefore

$$\begin{aligned} |(a - b_1)(x_k)| &\geqslant |(a - b)(x_k)| - |(b - b_1)(x_k)| \\ &> d - d/2(n + 1) - nd/2(n + 1) \\ &= d/2. \end{aligned}$$

The trouble with this argument is that it appears to depend on our prior knowledge of b, the very quantity we are trying to compute. Fortunately, there is a practical method of starting the Remes algorithm, as we now show.

Define $\xi_k = (k-1)/n$ $(k = 1, \ldots, n+1)$, and construct $b_0 \in H$ so that

$$\begin{aligned} |(a-b_0)(\xi_k)| &= d_0 \\ &= \inf_{h \in H} \max_{r=1,\ldots,n+1} |(a-h)(\xi_r)| \end{aligned}$$

for each $k \in \{1, \ldots, n+1\}$ [4, 5.1]. Let ω be a modulus of continuity for $a - b_0$ on $[0, 1]$, $t = \min(n^{-1}, \omega(d/2))$, $c = (\gamma(t)/\|\phi\|)^n / \sum_{k=1}^{n+1} (\|\phi\|/\gamma(t))^{k-1}$. Either $d_0 > 0$, in which case we need only set $x_{1,k} = \xi_k$ for each k; or, as we may now suppose, $d_0 < cd/4$. Choose $\zeta \in [0, 1]$ with $|(a-b_0)(\zeta)| > 3d/4$. Then $|\zeta - \xi_k| \geqslant \omega(d/2)$ for each k. Thus there exists r with $\xi_r < \zeta < \xi_{r+1}$. Define $x_{1,r} = \zeta$, $x_{1,k} = \xi_k$ $(k \in \{1, \ldots, n+1\},\ k \neq r)$. We show that the sequence $(x_{1,1}, \ldots, x_{1,n+1})$ will start the Remes algorithm.

Choose $j \in \{0, 1\}$ so that $(-1)^{r+j}(a-b_0)(x_{1,r}) > 3d/4$. Let $\lambda_1, \ldots, \lambda_{n+1}$ be real numbers such that $(-1)^{1+j}\lambda_1 > 0$, $\sum_{i=1}^{n+1} |\lambda_i| = 1$ and $\sum_{i=1}^{n+1} \lambda_i \phi(x_{1,i}) = 0$. Noting that $\min_{k=1,\ldots,n}(x_{1,k+1} - x_{1,k}) \geqslant t$ and referring to 2.1, we obtain

$$\begin{aligned} \sum_{i=1}^{n+1} \lambda_i a(x_{1,i}) &= \sum_{i=1}^{n+1} \lambda_i (a-b_0)(x_{1,i}) \\ &= \sum_{i=1}^{n+1} |\lambda_i| \, (-1)^{i+j}(a-b_0)(x_{1,i}) \\ &> \sum_{i=1, i\neq r}^{n+1} |\lambda_i| \, (-d_0) + |\lambda_r| \, 3d/4 \\ &\geqslant -d_0 + 3cd/4 \\ &> cd/2. \end{aligned}$$

Hence, by 2.4,

$$\begin{aligned} 0 &< \sum_{i=1}^{n+1} \lambda_i a(x_{1,i}) \\ &= \inf_{h \in H} \max_{k=1,\ldots,n+1} |(a-h)(x_{1,k})|, \end{aligned}$$

as we required.

Thus we see that the Remes algorithm can be started after at most a preliminary "single exchange."

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